

# CARTOGRAPHER: a tool for string diagrammatic reasoning

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## Abstract

We introduce CARTOGRAPHER, a tool for editing and rewriting string diagrams of symmetric monoidal categories. Our approach is principled: the layout exploits the isomorphism between string diagrams and certain cospans of hypergraphs; the implementation of rewriting is based on the soundness and completeness of convex double-pushout rewriting for string diagram rewriting.

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## 1 Introduction

String diagrammatic theories are increasingly important in computer science. They have been recently been used in a number of applications, including enabling the simplification of quantum circuits using the ZX-calculus [10], compositional descriptions of models of concurrency such as Petri Nets [18, 6], compositional accounts of signal flow graphs in control theory [7, 11, 1] and Bayesian reasoning [8, 14, 13]. These examples, as well as many others, work with the language of *symmetric monoidal categories* (SMCs). This paper addresses the need for tool support for *symmetric monoidal theories* - graphical rewriting systems of SMCs.

CARTOGRAPHER is a graphical editor and proof assistant for symmetric monoidal theories. It provides a graphical string diagram editor to construct morphisms, and a prover in which rewrite rules can be specified and executed. Further, CARTOGRAPHER has a firm theoretical foundation, its rewriting backend based on recent work in the area [5, 3, 20, 4]. The goal of this paper is to motivate CARTOGRAPHER, explain the basic features of the backend and the front end, and describe some of the technical challenges that were solved in creating it. The tool and its user guide are available on the CARTOGRAPHER website at <http://cartographer.id/>.

Our motivating example is the rewriting system in Figure 1. The intended semantic interpretation is that of binary circuits, where each wire carries an  $n$  bit number for some fixed  $n$ . Green nodes with two outputs copy numbers, those with no outputs discard their input, while red nodes perform addition modulo  $2^n$ .



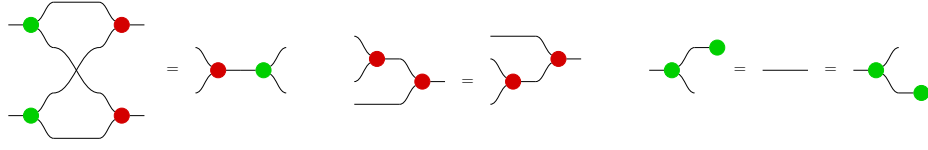
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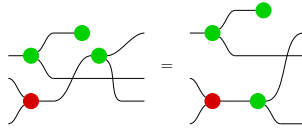
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■ **Figure 1** Example rules for binary circuits with copying ( $\text{---}\bullet\text{---}$ ), adding ( $\text{---}\bullet\text{---}$ ), and discarding ( $\text{---}\bullet$ ).

As well as the rules in Figure 1, this rewriting system implicitly uses three *generators*; atomic sub-diagrams, each with some number of inputs and outputs. These are the *copy* ( $\text{---}\bullet\text{---}$ ), *add* ( $\text{---}\bullet\text{---}$ ), and *discard* ( $\text{---}\bullet$ ) operations. The laws of symmetric monoidal categories permit moving generators around up to an isotopy made precise in [15, 19]. For our purposes, it suffices to say informally that generators can be slid along wires, and moved around on the page, but not rotated. By way of example, consider the equivalent diagrams in Figure 2.



■ **Figure 2** Example of string diagrams considered equal under the laws of SMCs

CARTOGRAPHER allows reasoning modulo the laws of symmetric monoidal categories. The user can deform morphisms up to the SMC laws without making proofs unsound, and the prover does not require (e.g. when matching the l.h.s. of a rule) the user to explicitly use the laws of symmetric monoidal categories. Put another way, the user should not have to “untangle” the wires of the diagram before applying a rule of some theory.

To put this into context, compare CARTOGRAPHER to two “competing” tools: Quantomatic [17] and Globular [2] (or its more recent descendant, `homotopy.io`). In a sense, CARTOGRAPHER sits between them: providing a more general setting than Quantomatic, while at the same time being more focussed than Globular.

software	generality	geometric intuition
Quantomatic	compact-closed	generators can implicitly be moved and wires bent back
Cartographer	symmetric monoidal	generators can implicitly be moved
Globular	higher categories	no implicit deformations permitted

Quantomatic deals with *compact closed* categories, in which not only may generators be moved, but wires may be “bent backwards”. In terms of our circuit analogy, this would allow for feedback, e.g. as used in a simple latch. CARTOGRAPHER allows such feedback, but as an explicit compact closed structure in the theory at hand, not implicitly assumed to exist by the underlying tool. On the other hand, Globular is much more general, aiming to support diagrammatic reasoning in higher categories. While this allows more freedom, when working with SMCs it comes at the cost of having to explicitly use SMC laws in proofs, e.g. using the functoriality of the monoidal product to slide two generators past each other.

## Contributions

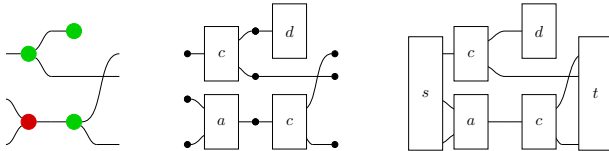
The contribution of CARTOGRAPHER is twofold. First, in the back end we implement an algorithm for matching and rewriting modulo the laws of SMC based on the adequacy result of [5]. The algorithm works with a data structure for *Open Hypergraphs*, which we introduce in this paper. Second, in the front end, we use an algorithm for the layout of these directed acyclic open hypergraphs which behaves well under rewriting and deformation of diagrams.

## 2 Directed Acyclic Open Hypergraphs

The main problem of implementing rewriting modulo symmetric monoidal structure is in finding a data structure in which equivalent terms have a single representation. For example, the two equivalent diagrams of Figure 2 should have the same underlying representation. Our approach is principled, because it uses the isomorphism between equivalent terms and cospans of hypergraphs found in [5]. Starting from this result, we propose an alternative but equivalent representation which is more convenient to work with.

We begin with an overview of the open hypergraphs of [5], and the CARTOGRAPHER data structure—illustrated in Figure 3 along with the corresponding string diagram. Beginning with the central (open) hypergraph, hyperedges are denoted  $\boxed{\phantom{a}}$ , and represent the generators of the string diagram. There are two kinds of nodes, both denoted  $\bullet$ . Firstly (ordered) boundary nodes, which are connected to a single wire (input or output, but not both). Secondly, internal nodes, having exactly one input and one output wire. These two conditions are the “monogamcity” requirement of [5], and effectively ensure that the hypergraph corresponds to a string diagram. For full details, see [5, Definition 3.6].

In contrast, the hypergraphs of CARTOGRAPHER are closed, and so nodes are rendered simply as wires, each with exactly one input and one output connection. Boundary nodes are replaced by adding special generators to the signature of the hypergraph,  $s$  (boundary source) and  $t$  (boundary target). Nodes are then uniquely identified by the two “ports” they connect—a *port* being a specific position on the boundary of a hyperedge.



**Figure 3** From left to right: a string diagram, its open hypergraph representation with signature  $\Sigma = \{a, c, d\}$ , and the equivalent closed hypergraph with signature  $\Sigma' = \Sigma \cup \{s, t\}$

► **Definition 1.** A  $k \rightarrow m$  CARTOGRAPHER *hypergraph*  $(\Sigma, E, W)$  consists of:

- the *signature*  $\Sigma$ , which can be thought of as the set of *types* of hyperedges. Each has arity  $ar : \Sigma \rightarrow \mathbb{N} \times \mathbb{N}$ , giving the number of inputs and outputs. We require that the  $\Sigma$  contains *boundary* generators  $\sigma, \tau$ , with  $ar(\sigma) = (0, k)$  and  $ar(\tau) = (m, 0)$ ;
- the set of *hyperedges*  $E$ , with a function  $typ : E \rightarrow \Sigma$  that assigns types to hyperedges. Moreover, there are *boundary* hyperedges  $\{s, t\} \subseteq E$  s.t.  $typ^{-1}(\sigma) = \{s\}$ ,  $typ^{-1}(\tau) = \{t\}$ ;
- the set of *wires*  $W$ . Given a hyperedge  $e \in E$ , if  $ar(typ(e)) = (p, q)$  then we say  $e$  has  $p$  input ports, denoted  $e^1, e^2, \dots, e^p$ , and  $q$  output ports denoted  $e_1, e_2, \dots, e_q$ . A *wire*  $w \in W$  is an ordered pair  $(e_i, f^j)$  of a *source port*  $e_i$  and a *target port*  $f^j$ , denoting a directed connection from the  $i^{th}$  output of  $e$  to the  $j^{th}$  input of  $f$ .

### 3 Visualising and Editing Open Hypergraphs

In contrast to Quantomatic [17] which uses a force-directed layout, and Globular [2] which has a fixed style for morphism layout, we use a *layered graph drawing* algorithm similar to that of Dot [12]. Our reasons for choosing layered graph drawing are as follows. Firstly, it was an aesthetic choice to represent string diagrams similarly to how they appear in the literature. Secondly, string diagrams drawn with the layered discipline retain a closer link with the underlying algebraic description of morphisms, since the term can be easily read off the string diagram in the form of a composition-of-monoidal-products. Thirdly, in contrast to *force-directed* approaches, the elements of a layered graph layout do not move around on the page, which is problematic from a user-experience perspective, because they are harder for the user to click. Additionally, force-directed layouts can change significantly after a rewrite rule is applied, with little control over the resulting diagram. This can be confusing for the user, because the string diagram may look very different. Finally, using layered hypergraphs offers a simple and intuitive way to enforce acyclicity: users may only connect generators if the target appears to the right of the source.

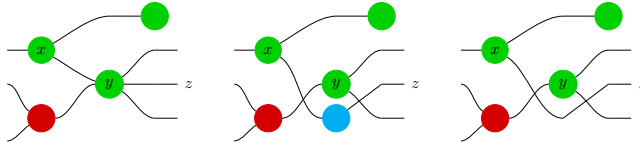
#### Interactive Layered Graph Drawing

We briefly summarise the interactive layered graph drawing approach of CARTOGRAPHER. By “interactive”, we mean to distinguish CARTOGRAPHER’s layout algorithm from other layered graph drawing approaches—such as Dot’s—in which a static graph is given as input, and positions of nodes and edges are returned. CARTOGRAPHER allows for the *incremental construction* of hypergraphs, meaning that users begin with a blank canvas, and add generators and connections one-by-one. We call it a layered graph drawing approach because it uses two key ideas from those approaches: the user of *layers*, and of *pseudonodes*.

► **Definition 2.** Given a CARTOGRAPHER hypergraph  $(\Sigma, E, W)$  and  $e \neq e' \in E$ , there is a directed path from  $e$  to  $e'$  if there exists a sequence  $(e_1, \dots, e_n)$  where  $e_i \in E$ ,  $e_1 = e$ ,  $e_n = e'$  and for each  $e_i, e_{i+1}$  there exist  $j_1, j_2$  such that  $((e_i)_{j_1}, (e_{i+1})^{j_2}) \in W$ . A *layering* is a function  $L : E \rightarrow \mathbb{N}$  such that:

- (i) if there is a directed path from  $e$  to  $e'$  then  $L(e) < L(e')$ ;
- (ii) for every non-boundary hyperedge  $e \in E$ ,  $L(s) < L(e) < L(t)$ .

The layering  $L$  essentially serves as the “ $x$  coordinate” of each hyperedge. The second idea from layered graph layout is the use of *pseudonodes*, which are conceptually related to the edge-points of Dixon and Kissinger’s Open Graphs [9], but used here only for layout purposes: they prevent wires from crossing generators. For a concrete example of why this is desirable, consider Figure 4. In the left-hand diagram, the wire from  $x$  to  $z$  passes through  $y$  and it is not clear whether  $x$  is connected to  $y$  and  $y$  to  $z$ , or if  $x$  is directly connected to  $z$ . Inserting pseudonodes into the graph clears up the ambiguity.



■ **Figure 4** Left, a diagram with only generators (rendered ● and ●), center, the same diagram after inserting pseudonodes (rendered ●), and right, the diagram as it appears with pseudonodes hidden.

## 136 The Layout Algorithm

137 We briefly outline the layout algorithm used in CARTOGRAPHER . Because the algorithm is  
 138 interactive, it takes the form of a *layout state*, and a number of *actions* that the user can  
 139 take. We model these actions as functions of the layout state.

140 The layout state is a tuple  $(H, G)$  of a hypergraph  $H$  as in definition 1, and an *integer*  
 141 *grid*  $G$ , which keeps track of the positions of generators and pseudonodes as two dimensional  
 142 vectors. Users can perform two actions on the layout state:

- 143 1. Placing a generator at a specific position on an integer grid
- 144 2. Moving a generator from one position to another
- 145 3. Connecting a source port to a target port

146 Moving and placing a generator is straightforward: if a generator  $e$  is moved or placed  
 147 such that it would overlap with another generator  $f$ , then  $f$  is moved down within the same  
 148 layer to make space. However, when connecting ports we must ensure that the hypergraph  
 149  $H$  remains acyclic. This is enforced using the following constraints:

- 150 ■ If generators  $e, f$  have layers such that  $L(e) \leq L(f)$ , then outputs of  $f$  may not be  
 151 connected to inputs of  $e$ .
- 152 ■ If a generator  $f$  is reachable from  $e$ , then  $f$  may not be moved such that  $L(f) \leq L(e)$ .

153 These constraints ensure that layering respects the properties of Definition 2, preserving  
 154 acyclicity. Finally, for every operation, the set of required pseudonodes is maintained, along  
 155 with their positions in  $G$ . In particular, this means updates for any operation which changes  
 156 connectivity, or modifies the number of layers between two generators.

## 157 4 Matching, Convexity, Rewriting

158 As well as an interactive string diagram editor, CARTOGRAPHER enables diagrammatic  
 159 reasoning. A derivation consists of a series of rewrites, using a set of rules specified by the  
 160 user. A *rule* consists of two CARTOGRAPHER hypergraphs, the lhs and the rhs, with identical  
 161 boundaries. Rewriting is implemented by double-pushout rewriting of hypergraphs, with  
 162 soundness and completeness guaranteed by [5, Theorem 5.6].

163 Applying a rule to a string diagram consists of three steps: finding a match for the lhs a rule,  
 164 checking for convexity, and applying the rewrite rule. A *match* is an hypergraph embedding  
 165 (an injective, homomorphic mapping of hyperedges and nodes) of open hypergraphs, with  
 166 one subtlety: the boundary ports of the pattern match can map to non-boundary ports in  
 167 the target. CARTOGRAPHER builds matches incrementally by using the backtracking logic  
 168 library *logict* [16]. Roughly speaking, wires and generators are added to the working match  
 169 until either there are no more unmatched wires or generators, or a contradiction is reached,  
 170 in which case the search backtracks. Candidate matches are then checked for *convexity* [5],  
 171 which is needed for a rewrite to be valid modulo the laws of SMCs. Roughly speaking, all  
 172 directed paths that start and end in a matched region must remain within the match. Once a  
 173 convex match has been identified, the internal hyperedges of the matched region are removed  
 174 and replaced with the right hand side of the rewrite rule.

175 The CARTOGRAPHER UI shows a list of matches of rules found in the current proof term.  
 176 Users can apply rewrites by hovering over each match to see which part of the graph will be  
 177 rewritten, and then clicking to apply the rewrite.

## 5 Conclusions and Future Work

- CARTOGRAPHER is still in early stages of development. We are working on
- improving the layout algorithms by adapting heuristics from other tools that work with layered graphs;
  - more advanced features for diagrammatic reasoning, including support for structured proofs (using e.g. user-generated Lemmas) and adapting other user-friendly features originally developed for theorem provers and proof assistants;
  - higher level specification features, such as support for bang-boxes, recursive definitions, and proof strategies;
  - better decoupling between the rewriting back end and the layout front end, enabling extensions such as rewriting modulo compact closed structure.
  - optional non-convex rewriting, to allow for SMCs with built-in Frobenius structure.

## References

- 1 John Baez and Jason Erbele. Categories in control. *Theory and Applications of Categories*, 30:836–881, 2015.
- 2 Krzysztof Bar, Aleks Kissinger, and Jamie Vicary. Globular: an online proof assistant for higher-dimensional rewriting. In *Leibniz International Proceedings in Informatics*, volume 52, pages 34:1–34:11, 2016. [ncatlab.org/nlab/show/Globular](http://ncatlab.org/nlab/show/Globular).
- 3 Filippo Bonchi, Fabio Gadducci, Aleks Kissinger, Paweł Sobociński, and Fabio Zanasi. Confluence of graph rewriting with interfaces. In *Programming Languages and Systems - 26th European Symposium on Programming, ESOP 2017, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2017, Uppsala, Sweden, April 22-29, 2017, Proceedings*, pages 141–169, 2017. URL: [https://doi.org/10.1007/978-3-662-54434-1\\_6](https://doi.org/10.1007/978-3-662-54434-1_6), doi:10.1007/978-3-662-54434-1\_6.
- 4 Filippo Bonchi, Fabio Gadducci, Aleks Kissinger, Paweł Sobociński, and Fabio Zanasi. Rewriting with frobenius. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018*, pages 165–174, 2018. URL: <https://doi.org/10.1145/3209108.3209137>, doi:10.1145/3209108.3209137.
- 5 Filippo Bonchi, Fabio Gadducci, Aleks Kissinger, Paweł Sobociński, and Fabio Zanasi. Rewriting modulo symmetric monoidal structure. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science - LICS '16*, pages 710–719, New York, NY, USA, 2016. ACM Press. URL: <http://dl.acm.org/citation.cfm?doid=2933575.2935316>, doi:10.1145/2933575.2935316.
- 6 Filippo Bonchi, Joshua Holland, Robin Piedeleu, Paweł Sobociński, and Fabio Zanasi. Diagrammatic algebra: from linear to concurrent systems. *PACMPL*, 3(POPL):25:1–25:28, 2019. URL: <https://dl.acm.org/citation.cfm?id=3290338>.
- 7 Filippo Bonchi, Paweł Sobociński, and Fabio Zanasi. The calculus of signal flow diagrams I: linear relations on streams. *Inf. Comput.*, 252:2–29, 2017.
- 8 Benjamin Cabrera, Tobias Heindel, Reiko Heckel, and Barbara König. Updating probabilistic knowledge on condition/event nets using bayesian networks. In *29th International Conference on Concurrency Theory, CONCUR 2018, September 4-7, 2018, Beijing, China*, pages 27:1–27:17, 2018. URL: <https://doi.org/10.4230/LIPIcs.CONCUR.2018.27>, doi:10.4230/LIPIcs.CONCUR.2018.27.
- 9 Lucas Dixon and Aleks Kissinger. Open-graphs and monoidal theories. *Mathematical Structures in Computer Science*, 23(2):308–359, 2013.
- 10 Andrew Fagan and Ross Duncan. Optimising Clifford Circuits with Quantomatic. *Electronic Proceedings in Theoretical Computer Science*, 287:85–105, January 2019. URL: <http://arxiv.org/abs/1901.10114v1>, doi:10.4204/EPTCS.287.5.

- 226 11 Brendan Fong, Paolo Rapisarda, and Paweł Sobociński. A categorical approach to open and  
 227 interconnected dynamical systems. In *Thirty-first annual ACM/IEEE symposium on Logic  
 228 and Computer Science (LiCS 2016)*, pages 495–504, 2016. doi:10.1145/2933575.2934556.
- 229 12 E.R. Gansner, E. Koutsofios, S.C. North, and K.-P. Vo. A technique for drawing directed  
 230 graphs. *IEEE Transactions on Software Engineering*, 19(3):214–230, March 1993. URL:  
 231 <http://ieeexplore.ieee.org/document/221135/>, doi:10.1109/32.221135.
- 232 13 Bart Jacobs, Aleks Kissinger, and Fabio Zanasi. Causal inference by string diagram surgery.  
 233 2019. URL: <http://arxiv.org/abs/1811.08338>.
- 234 14 Bart Jacobs and Fabio Zanasi. The logical essentials of bayesian reasoning. In Joost-Peter Ka-  
 235 toen Gilles Barthe and Alexandra Silva, editors, *Probabilistic Programming*. Cambridge  
 236 University Press, Cambridge, 2019. URL: <http://arxiv.org/abs/1804.01193>.
- 237 15 Andre Joyal and Ross Street. The geometry of tensor calculus, I. *Adv. Math.*, 88:55–112, 1991.
- 238 16 Oleg Kiselyov, Chung-chieh Shan, Daniel P Friedman, and Amr Sabry. Backtracking, Inter-  
 239 leaving, and Terminating Monad Transformers. page 12.
- 240 17 Aleks Kissinger and Vladimir Zamdzhiev. Quantomatic: A Proof Assistant for Diagrammatic  
 241 Reasoning. *arXiv:1503.01034 [cs, math]*, 9195:326–336, 2015. arXiv: 1503.01034. URL:  
 242 <http://arxiv.org/abs/1503.01034>, doi:10.1007/978-3-319-21401-6\_22.
- 243 18 José Meseguer and Ugo Montanari. Petri nets are monoids. *Information and Computa-  
 244 tion*, 88(2):105–155, October 1990. URL: [https://linkinghub.elsevier.com/retrieve/  
 245 pii/0890540190900138](https://linkinghub.elsevier.com/retrieve/pii/0890540190900138), doi:10.1016/0890-5401(90)90013-8.
- 246 19 Peter Selinger. A survey of graphical languages for monoidal categories. *arXiv:0908.3347  
 247 [math]*, 813:289–355, 2010. arXiv: 0908.3347. URL: <http://arxiv.org/abs/0908.3347>,  
 248 doi:10.1007/978-3-642-12821-9\_4.
- 249 20 Fabio Zanasi. Rewriting in free hypergraph categories. In *Proceedings Third Workshop on  
 250 Graphs as Models, GaM@ETAPS 2017, Uppsala, Sweden, 23rd April 2017.*, pages 16–30, 2017.  
 251 URL: <https://doi.org/10.4204/EPTCS.263.2>, doi:10.4204/EPTCS.263.2.